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Ranking Theory and Conditional Reasoning

Niels Skovgaard-Olsen[a,b]

[a]*Philosophy Department, University of Konstanz*
[b]*Psychology Department, University of Freiburg*

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Abstract

Ranking theory is a formal epistemology that has been developed in over 600 pages in Spohn's recent book *The Laws of Belief*, which aims to provide a normative account of the dynamics of beliefs that presents an alternative to current probabilistic approaches. It has long been received in the AI community, but it has not yet found application in experimental psychology. The purpose of this paper is to derive clear, quantitative predictions by exploiting a parallel between ranking theory and a statistical model called logistic regression. This approach is illustrated by the development of a model for the conditional inference task using Spohn's (2013) ranking theoretic approach to conditionals.

Keywords: Ranking theory; Conditional reasoning; Reasons; Psychology of reasoning; Relevance; Semantics of conditionals; Quantitative predictions

1. Introduction

Recently, there has been a paradigm-shift in the psychology of reasoning (Evans, 2012). Whereas earlier research used deductive logic as the main normative model, recent research has started to use probabilistic, Bayesian models, which represent our degrees of beliefs as subjective probabilities. In the study of conditionals this is seen by the current popularity of the suppositional theory of conditionals, which is based on the Ramsey test. The Ramsey test consists of adding the antecedent to one's knowledge base, while making as few changes as possible, and evaluating the consequent on its supposition. This theory thus models our understanding of the indicative, natural language conditional as a conditional probability (Bennett, 2003; Edgington, 1995; Evans & Over, 2004; Oaksford & Chater, 2007, ch. 5; Oaksford & Chater, 2010a).

Correspondence should be sent to Niels Skovgaard-Olsen, Gießenstr. 18, 4. Stockwerk, 79104 Freiburg, Herdern, Germany. E-mails: niels.skovgaard.olsen@psychologie.uni-freiburg.de and n.s.olsen@gmail.com

One issue that has been discussed extensively in formal epistemology, however, is the problem of how to formally represent not degrees of beliefs but full beliefs in light of the lottery paradox, which poses difficulties for the intuitive idea that full beliefs consist of probabilities above a certain threshold (cf. Huber, 2013).[1]

Explication of the lottery paradox: suppose that Bel(A) iff $P(A) > \alpha$, where Bel(A) represents a full belief in A. If a fair lottery is held with $> \alpha^{-1}$ tickets and exactly one winner, then the probability that each individual ticket won't win is above our threshold. Thus, we will believe of each individual ticket that it won't win. Yet, the conjunction of each of these beliefs is inconsistent with the belief that one of the tickets will win. Therefore, we apparently cannot both hold that beliefs are closed under conjunctions and accept that we have full beliefs in propositions that are assigned a probability above α. However, in formal epistemology the closure of (rational) beliefs under logical operations is taken to be a minimal requirement of rationality (cf. Skovgaard-Olsen (in review)).

Ranking theory was developed as a unified framework for representing both degrees of beliefs and full beliefs that avoids this problem. In Spohn (2012) it has, moreover, been elaborated into a comprehensive formal epistemology that is able to provide a normative account of the dynamics of beliefs and non-monotonic reasoning.

As Leitgeb (2007) argues, it has not yet made it into the common knowledge of cognitive scientists, but when theorizing about conditionals it is useful to invoke a notion of *conditional beliefs* that is to be conceptually distinguished both from *beliefs in conditionals* and from *unconditional beliefs in propositions*. In contrast with unconditional beliefs, conditional beliefs do not themselves take 'true' and 'false' as values. Rather, they are *bi-propositional attitudes*[2] that connect two propositional contents $\{A,C\}$ without themselves expressing a proposition. Such bi-propositional attitudes are, *inter alia*, manifested in inferential dispositions to infer C from A (in the absence of disabling conditions). Furthermore, the impossibility results of Lewis and Gärdenfors cited in Leitgeb (2007) prevent their identification with beliefs in conditionals. Thus, conditional beliefs are a distinct species in our mental architecture that require a separate investigation (see also Spohn, 2012, section 9.1).

Spohn (2013) argues that the problems with representing full (unconditional) beliefs in probabilistic terms carry over to the treatment of conditionals, insofar as these problems also affect this notion of conditional beliefs. The point is that the following logical law of indicative conditionals:

if 'if A, B' and 'if A, C' then 'if A, $B \cap C$'

has its basis in the following law of rational, conditional belief:

if Bel($B|A$) and Bel($C|A$) then Bel($B \cap C|A$)

where the conditional belief in B, given A, is represented as Bel($B|A$). Yet, an identification of conditional beliefs with high conditional probabilities would not be able to secure this latter law, because the problems posed by the lottery paradox generalize to the conditional case (ibid.: 1081–82). Consequently, Spohn (2013) uses ranking theory to state the semantics of conditionals. Its principal advantage consists in possessing a notion of conditional beliefs that steers free of such troubles. A further benefit of the theory is that it makes formally precise the old idea, not captured in the suppositional theory, that the antecedent should somehow be relevant to the consequent in natural language conditionals, as we shall see.

Ranking theory has already been received in the AI community (cf. Goldszmidt & Pearl, 1996; Kern-Isberner & Eichhorn, 2014). However, its application in psychology is still to come. One obstacle delaying this has been difficulties in deriving clear, experimentally distinguishable predictions. As we shall see, the theory of conditionals presented in Spohn (2013) provides some qualitative predictions. But it is not clear how to turn these into precise, quantitative predictions. The extension of ranking theory to be presented in section 3 improves the situation.

2. Ranking theory

Before we dwell on these topics, it will serve our purposes, if we first have a simple presentation of ranking theory. Ranking theory is built up on a metrics of beliefs, which quantifies a grading of *disbelief* expressed by *negative* ranking functions, κ. The object of our degrees of disbelief is taken to be propositions (i.e., the content shared by sentences of different languages). To formally represent propositions, ranking theory follows possible world semantics in representing propositions as sets of "possible worlds" or possibilities. Therefore, to state ranking theory, first a non-empty set, W, of mutually exclusive and jointly exhaustive possibilities is assumed. Next, an algebra, $\mathcal{A}$, of subsets of W is formed so that it is closed under logical operations. This collection of subsets of W represents all possible propositions. Doxastic attitudes such as *believing* and *disbelieving* propositions can then be represented by functions that are defined over $\mathcal{A}$. Accordingly, negative ranking functions expressing an agent's degrees of disbelief can now be defined as follows:

Definition 1: let $\mathcal{A}$ be an algebra over W. Then, κ is a *negative ranking function* for $\mathcal{A}$ iff κ is a function from $\mathcal{A}$ into $N \cup \{\infty\}$, the set of natural numbers plus infinity, such that for all $A, B \in \mathcal{A}$:

$$\kappa(W) = 0 \text{ and } \kappa(\emptyset) = \infty \tag{1}$$

$$\kappa(A \cup B) = \min\{\kappa(A), \kappa(B)\}, \tag{2}$$

where $\kappa(A)$ is called the *negative rank* of A. From the above it follows that:

$$\kappa(A) = 0 \text{ or } \kappa(\bar{A}) = 0 \text{ or both} \tag{3}$$

If $\kappa(A) < \infty$, then the *conditional rank* of B, given A, is defined as follows:

$$\kappa(B|A) = \kappa(A \cap B) - \kappa(A) \tag{4}$$

The intuition here is that the degree of disbelief in B, given A, is evaluated by considering the degree of disbelief in $A \cap B$ while removing the disbelief in A (insofar as B is now to be evaluated under the supposition of A).

Since negative ranks are said to represent degrees of disbelief, $\kappa(A) = 0$ represents that A is *not disbelieved*. When $\kappa(A)$ assigns a value of $n > 0$ to A, then A is said to be disbelieved to the nth degree. *Doxastic indifference* is represented by neither disbelieving A nor $\sim A$, that is, $\kappa(A) = \kappa(\bar{A}) = 0$, and *belief in A* is represented in an indirect way by disbelief in $\sim A$, that is, $\kappa(\bar{A}) > 0$. Moreover, conditional ranks are used to represent conditional beliefs.

On this basis, *positive ranking functions* representing degrees of beliefs can be defined for A by:

$$\beta(A) = \kappa(\bar{A}) \tag{5}$$

Positive ranking functions can then be axiomatized by translating Eqs. 1, 2, and 4 into their positive equivalents:

$$\beta(W) = \infty \text{ and } \beta(\emptyset) = 0 \tag{6}$$

$$\beta(A \cap B) = \min\{\beta(A), \beta(B)\} \tag{7}$$

$$\beta(B|A) = \beta(\bar{A} \cup B) - \beta(\bar{A}) \tag{8}$$

where the conditional degree of belief in B, given A, is represented as the degree of belief in the material implication ($A \supset B \equiv \sim A \cup B$) while subtracting the degree of belief in $\sim A$, which is the false-antecedent case, where the material implication is satisfied trivially.

Moreover, it is possible to define two-sided ranking functions for A that combine the gradings of disbelief and belief into one function:

$$\tau(A) = \beta(A) - \kappa(A) = \kappa(\bar{A}) - \kappa(A) \tag{9}$$

$$\tau(B|A) = \beta(B|A) - \kappa(B|A) = \kappa(\bar{B}|A) - \kappa(B|A) \tag{10}$$

2.1. Ranking theory and the probability calculus

When one compares formal epistemologies expressed by ranks and probabilities, it becomes apparent that there is a deep parallel between negative ranking functions and probability distribution functions, as exhibited in Table 1 below (Spohn, 2012, ch. 5).

This is no accident. As Spohn (2009) points out, probabilities can be translated into negative ranks. By applying the translation manual below, one is *almost* sure to obtain a ranking theorem from any probabilistic theorem:

> There is obviously a simple translation of probability into ranking theory: translate the sum of probabilities into the minimum of ranks, the product of probabilities into the sum of ranks, and the quotient of probabilities into the difference of ranks. Thereby, the probabilistic law of additivity turns into the law of disjunction, the probabilistic law of multiplication into the law of conjunction (for negative ranks), and the definition of conditional probabilities into the definition of conditional ranks. If the basic axioms and definitions are thus translated, then it is small wonder that the translation generalizes; take any probabilistic theorem, apply the above translation to it, and you almost surely get a ranking theorem. (p. 209)

If negative ranking functions are treated as the logarithms of probabilities with a base, $a \in (0,1)$, the translation is captured of products and quotients of probabilities as the sum and difference of ranking functions. However, Spohn (ibid., 2012, p. 203) points out that if the sum of probabilities is to be translated into the minimum of ranks, the logarithmic base has to be infinitesimal. Therefore, for purposes of theoretical unification the latter translation seems superior. Yet for psychological purposes this translation is deeply problematic, insofar as it would require that the participants had all of their degrees of disbelief in A expressed in probabilities from 0 to $0 + \varepsilon$, where ε is an infinitesimal quantity that is bigger than but arbitrarily close to zero.

Consequently, we will work with a logarithmic base that allows the degrees of beliefs to spread out more evenly across the probability scale (cf. section 7). But we must not forget that it comes with the prize of being unable to translate the sum of probabilities into the minimum of ranks, as noted above. This leaves us one step further away from a theoretical unification of ranks with probabilities, and it implies that we are only dealing with an approximation. However, this does not mean that the translation would have been perfect on the infinitesimal

Table 1
Comparison between the probability calculus and ranking theory

Probability Calculus	Ranking Theory
$P(A \cap B) = P(A) \cdot P(B \mid A)$	$\kappa(A \cap B) = \kappa(A) + \kappa(B \mid A)$
$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}$	$\kappa(A \mid B) = \kappa(B \mid A) + \kappa(A) - \kappa(B)$
$P(B) = \sum_{i}^{n} P(B \mid A_i) \cdot P(A_i)$	$\kappa(B) = \min_{i \leq n}[\kappa(B \mid A_i) + \kappa(A_i)]$

alternative. In fact, Spohn (2012: 204) already lists 12 deviations. Moreover, taking this step allows us to establish a connection with existing psychological literature.

However, it should be noted that from a psychological perspective, one general advantage of working directly with logarithms over probabilities is that the computations become much easier as difficult multiplications and divisions are now replaced by addition and subtraction. Hence, the more direct route of applying ranking theory directly to psychological experiments could also have been pursued. Indeed, in Juslin, Nilsson, Winman, and Lindskog (2011), some preliminary evidence has already been reported wherein the notorious base-rate neglect can be reduced once the task is presented in a logarithmic format instead of in a probabilistic format. Juslin et al. (2011) thus speculate that a linear, additive integration of information is the intuitive, default approach when people lack access to, or are unable to implement, overriding analytic (e.g., multiplicative) rules. Still, the present paper adopts the more conservative approach of first translating ranks into probabilities and then identifying applications to psychology of reasoning to establish more contact with existing research.

2.2. *Conditionals and reason relations*

In Spohn (2012, ch. 6) an epistemic notion of relevance is introduced, which is given a pivotal role in his semantics of conditionals (2013) and account of causation (2012, ch. 14). Inspired by the notion of statistical dependency and independency, Spohn defines relevance as follows:

$$A \text{ is positively relevant to } C \text{ iff} \qquad \tau(C|A) > \tau(C|\bar{A}) \tag{11}$$

$$A \text{ is irrelevant to } C \text{ iff} \qquad \tau(C|A) = \tau(C|\bar{A}) \tag{12}$$

$$A \text{ is negatively relevant to } C \text{ iff} \qquad \tau(C|A) < \tau(C|\bar{A}) \tag{13}$$

Furthermore, Spohn argues that this notion can be used to analyze the notion of reasons by holding that A is a reason *for* C iff Eq. 11 holds and a reason *against* C iff Eq. 13 holds. He is then able to use this notion of reasons to analyze four types of reason relations:

$$\text{Supererogatory reason} \qquad \tau(C|A) > \tau(C|\bar{A}) > 0 \tag{11a}$$

$$\text{Sufficient reason} \qquad \tau(C|A) > 0 \geq \tau(C|\bar{A}) \tag{11b}$$

$$\text{Necessary reason} \qquad \tau(C|A) \geq 0 > \tau(C|\bar{A}) \tag{11c}$$

$$\text{Insufficient reason} \qquad 0 > \tau(C|A) > \tau(C|\bar{A}) \tag{11d}$$

The starting point for Spohn's semantics is that conditionals express conditional beliefs, or some feature about our conditional beliefs. In ranking theory, our conditional beliefs can be expressed by $\tau(C|A) > 0$, and the inequalities above thus express features of our conditional beliefs. Now, in adopting this starting point, the most basic idea is that conditionals directly express conditional beliefs. Spohn (2015, p. 2) suggests that this is what the Ramsey test really amounts to. But as Spohn (2013) further argues, there are many other aspects of our conditional beliefs that conditionals can be used to express. In particular, conditionals can be used to express the reason relations described by Eqs. 11–13.[3] Whereas these inequalities focus our attention on the extent to which the antecedent is rank (or probability) raising for the consequent, the Ramsey test merely consists of adding the antecedent to our knowledge base and evaluating the probability of the consequent on its basis.

Thus, these constructions hold the promise of making the old idea precise that conditionals codify inferential relations and that the antecedent can be seen as a reason for the consequent in central applications of conditionals. As such, the ranking theoretic approach to conditionals finds itself in continuity with Ryle (1950), Rott (1986), Strawson (1986), Brandom (1994), Douven (2008, 2013), and Krzyżanowska (2015). A guiding idea of this tradition is that it constitutes *a semantic defect* when the antecedent of a conditional is irrelevant to the consequent, as in Edgington's (1995, p. 267) example: "If Napoleon is dead, Oxford is in England." In contrast, other accounts will have to set such infelicities aside as pragmatic phenomena that arise due to violations of Gricean norms of informative conversations. Yet, as Skovgaard-Olsen (forthcoming) argues, such an explanation suffers from the problem that these conditionals lack a standard interpretation, which can be decoded by a minimum of contextual information, whereby they would come out as felicitous even when dealing with individual reasoning. Hence, as they are both defective w.r.t. contexts of conversation and contexts of individual reasoning, their defect must be located in problems associated with their meaning.

A further example of a conditional connective that can be analyzed by means of Eqs. 11a–11d is "even if A, then still C." In such cases, the speaker may be seen as commenting on the assumption that A is a *reason against* C (which is made either by the speaker, hearer, or somebody else) and denying that A is *an insufficient reason against* C (ibid.). Moreover, Spohn (2015) lists other candidates for analysis through his account of reason relations such as "although," "despite," and "because." Accordingly, "C although A" roughly expresses that C was not to be expected, given A, "C despite A" likewise expresses negative relevance, and "C because A," *inter alia*, expresses that C was bound to obtain, given A. Finally, this list has been extended to cover about 30 further utterance modifiers such as "however" and "be that as it may" in Skovgaard-Olsen (forthcoming).

3. Extending ranking theory by logistic regression

The purpose of section 3 is to extend the ranking theoretic approach to conditionals by logistic regression to enhance the former's use for experimental psychology. In doing so, the present model follows an old, venerable tradition in cognitive psychology of using an analogy between statistics and cognition in formulating new theories. A good example is the signal detection theory, which arose through an analogy between sensory discrimination and hypothesis testing for statistical significance that led to the discovery of a new kind of data based on the idea of type I error (rejecting H_0 when in fact it is true) and type II error (failing to reject H_0 when in fact it is false) (see Gigerenzer & Murray, 1987 and Gigerenzer, 1988 for a detailed discussion).

As we shall see, the model to be introduced has the following nice qualities: (a) it provides equations that can produce quantitative predictions for the conditional inference task with only three parameters that are qualitatively constrained; (b) it throws new light on what is involved in performing the Ramsey test; (c) it allows us to introduce a numerical/verbal scale for two-sided ranking functions that has already found some empirical support; and (d) it allows us to combine a theory of conditionals that is already well-established in the psychology of reasoning with accounts emphasizing probability raising.

3.1. Logistic regression

In logistic regression, the probability that an observed, or measured, dependent variable, Y, takes the value “1” (i.e., “true”) is represented as a weighted, non-linear function of the values of a set of independent variables $\{X_1,\ldots,X_n\}$ that function as predictors and the intercept, b_0:

$$P(Y=1|X_1,\ldots,X_n)=\frac{z}{1+z}\quad \text{for}\, z=e^{b_0+b_1\cdot X_1+b_2\cdot X_2+\ldots+b_n\cdot X_n} \tag{14}$$

The indexed weights $\{b_1,\ldots,b_n\}$ are estimated from the data and express how much the indexed predictor contributes to reducing the variance in the dependent variable, when optimization methods are used to fit the model to the data (Eid, Gollwitzer, & Schmitt, 2010, ch. 21). For our purposes, it is moreover pleasing to note that what is being estimated is *the conditional probability* that the dependent variable takes a particular value. The model is thereby able to come into contact with: (i) approaches to conditionals that emphasize probability raising (Douven, 2008, 2013; Spohn, 2013), which focuses on the relationship between $P(C|A)$ and $P(C|\bar{A})$, and (ii) the suppositional theory of conditionals (Evans & Over, 2004), which takes $P(\text{if } A, C) = P(C|A)$ as its point of departure.[4] To simplify the calculations, Eq. 14 can be transformed into Eq. 14a:

$$P(Y=1|X_1,\ldots,X_n)=\frac{1}{1+\frac{1}{z}} \tag{14a}$$

Furthermore, it should be noted that due to the non-additive and non-linear relationship between the independent variables and probabilities in Eq. 14, the effect of one independent variable (X_i) varies with the values of the other independent variables and the predicted probabilities. For this reason the effect of X_i cannot be fully represented by a single coefficient; instead, it has to be evaluated at a particular value, or set of values, which renders its interpretation cumbersome (Pampel, 2000, ch. 2).

It may therefore be useful to note that two transformations for Eq. 14 exist, where the effect of X_i can be summarized by a single coefficient. When Eq. 14 is stated in terms of *conditional odds* (O_i),[5] e^{b_i} represents the factor by which the odds are multiplied when X_i increases by one unit and all other variables are held constant:

$$O_i = e^{b_0 + b_1 \cdot X_1 + b_2 \cdot X_2 + \ldots + b_n \cdot X_n} = e^{b_0} \cdot e^{b_1 \cdot X_1} \cdot e^{b_2 \cdot X_2} \ldots e^{b_n \cdot X_n} \qquad \text{for } i = 1, \ldots, n \tag{14b}$$

Furthermore, when Eq. 14 is stated in terms of *logged odds*, b_i represents how much $\hat{Y}$ changes with a one unit change to the indexed predictor when all other variables are held constant:

$$\ln(O_i) = b_0 + b_1 \cdot X_1 + b_2 \cdot X_2 + \ldots + b_n \cdot X_n \qquad \text{for } i = 1, \ldots, n \tag{14c}$$

Eq. 14c parallels multiple linear regression. (However, the units have changed to logged odds.)

3.2. Logistic regression and ranking theory

At first glimpse it may seem puzzling what this statistical model has to do with ranking theory. But the relationship between the two will gradually unfold throughout this paper. The purpose of this subsection is to introduce some initial observations.

As Spohn (2012: p. 76) points out, although using two-sided ranking functions may be the most intuitive way of presenting ranking theory, there is no simple axiomatization of them. Furthermore, since two-sided ranking functions appear to be a derived notion that is ultimately to be defined in terms of negative ranking functions, he prefers that the latter as an epistemological tool. However, as we will begin to see, the former is attractive for psychological purposes. Moreover, as we will now see, two-sided ranking functions are not merely the derived notion that they appeared to be. In fact, they have their own interpretation.

Since logistic regression deals with *logits*, or *logged odds*, it is interesting to note that two-sided ranking functions give us a comparable metrics. However, the logarithmic bases differ and two-sided ranking functions are actually the logged odds of a proposition *not* taking the value "true":

$$\tau(A) = \beta(A) - \beta(\bar{A}) = \kappa(\bar{A}) - \kappa(A)$$

$$\kappa(A) \approx \log_a(P(X = 1)), \text{where } 0 < a < 1$$

$$\tau(A) \approx \log_a(P(X = 0)) - \log_a(P(X = 1)) = \log_a\left(\frac{P(X = 0)}{P(X = 1)}\right)$$

To understand why two-sided ranking functions take this form, it is useful to consider that:

$$\log_a\left(\frac{P(x)}{1-P(x)}\right) = -\log_a\left(\frac{1-P(x)}{P(x)}\right)$$

$$\log_{\frac{1}{a}}\left(\frac{1-P(x)}{P(x)}\right) = -\log_a\left(\frac{1-P(x)}{P(x)}\right)$$

Therefore,

$$\log_{\frac{1}{a}}\left(\frac{1-P(x)}{P(x)}\right) = \log_a\left(\frac{P(x)}{1-P(x)}\right)$$

So when two-sided ranking functions are the logged odds of a proposition *not* taking the value "true," with a logarithmic base of $a \in (0,1)$, they can always be rewritten as the logged odds of a proposition taking the value "true," with the logarithmic base of a^{-1}.

Thus, if a logarithmic base of e^{-1} is chosen for ranking functions, it is possible to bring the two formalisms into contact, because the logarithmic base of our regression equations is e. As we shall see, this observation will later prove to be crucial when we begin deriving predictions from a model based on logistic regression for conditional reasoning.

4. The conditional inference task

When it comes to producing predictions for the psychology of reasoning, it is important to consider existing experimental paradigms, because most of the psychology of reasoning is organized around a few experimental paradigms that have been studied extensively (Manktelow, 2012). We will therefore continue our investigation of the parallel between logistic regression and ranking theory by focusing on a particular experimental paradigm.

In the conditional inference task the participants are asked to rate the conclusions of the following four inferences: MP (*modus ponens*: $p \rightarrow q$, $p \therefore q$), MT (*modus tollens*: $p \rightarrow q$, $\neg q \therefore \neg p$), AC (*affirmation of the consequent*: $p \rightarrow q$, $q \therefore p$), and DA (*denial of the antecedent*: $p \rightarrow q$, $\neg p \therefore \neg q$). Of these, only MP and MT are classically valid if '$\rightarrow$' is read as the material implication. As Singmann and Klauer (2011) point out, this task has been given with two different types of instructions. In the old, deductive paradigm the participants were typically asked to provide binary responses to questions about the validity of the conclusion, given the premises, while ignoring their background knowledge. In contrast, in the new, Bayesian paradigm the participants are typically asked to assess how likely the conclusion is, given the premises, on a graded scale, while invoking their background knowledge. In the old paradigm, the frequency with which MP was endorsed tended to be 89%–100% with abstract material, while the equally valid MT was typically only endorsed in 40%–80% of the cases. Moreover, the logically invalid AC and DA were typically endorsed in 20%–75% of the cases (Evans & Over, 2004, p. 46). In the new paradigm, all four inferences are likewise endorsed but to different degrees. These are some of the key findings that have contributed to questioning the appropriateness of deductive logic as a normative model of human reasoning in the current rationality debates in cognitive psychology (ibid.; Oaksford & Chater, 2007; Manktelow, 2012).

However, it is far from obvious that AC and DA should be seen as general flaws of reasoning. After all, AC characterizes the type of abductive inference embodied in Bayes' theorem, where we reason from an effect back to its potential cause,[6] which is characteristic of scientific reasoning. Bayes' theorem expresses this type of reasoning by requiring that we update our degree of belief in a hypothesis after the confirmation of one of its predictions:

$$P(H|E) = \frac{P(E|H) \cdot P(H)}{P(E)} \leftrightarrow P(A|C) = \frac{P(C|A) \cdot P(A)}{P(C)} \tag{15}$$

To be sure, I. Douven (personal communication) rightly points out that not only inferences to the *best* explanation take the form of AC, but also inferences to the *worst* explanation. However, what the ranking theoretic approach to conditionals validates is that, as we shall see, given there are acceptable forward inferences from A to C, the premise C should also be viewed as raising the degree of belief in the conclusion A. Thus, it is not a blank check to reason from C to A in whatever outrageous conditionals we like for which it holds that we would not be prepared to reason from A to C.

Moreover, DA also has its justification in argumentative contexts when it is used to challenge a reason that has been offered in support of C by urging that C has been advanced on an insufficient basis, as Godden and Walton (2004) argue.

It would therefore seem premature to dismiss the endorsement of these types of inferences as a symptom of irrationality merely because such inferences are connected with uncertainty and fail to be validated by classical logic. Thus, it is an attractive feature of Spohn's (2013) relevance approach that it is not forced to render these inferences invalid. In fact, Spohn's (2013) theory entails that the respective premises in MP, MT, AC, and DA all strengthen the degree of belief in their respective conclusions. However, this does not yet imply that the premises constitute *sufficient reasons* for the conclusions and that the conclusions of AC and DA should thus be accepted in an all-or-nothing sense.[7] What it means is rather that (a) the premises provide support for the conclusions in all four inferences, (b) there will be instances of AC and DA where the conclusion should be rationally accepted in an all-or-nothing sense; even if this rule does not hold for all cases, and finally that (c) the conclusions of all four inferences should have a non-zero degree of endorsement. In contrast, the suppositional theory of conditionals renders AC and DA invalid (Evans & Over, 2004, p. 45).

Moreover, the ranking theoretic approach to conditionals is compatible with the asymmetry in the degrees of endorsement that has been found. However, it is unable to deliver any precise, quantitative predictions about these endorsement rates, as it stands. This, however, is accomplished by the extension of the theory to be presented below.

To back up a little, what leads to the claims above concerning AC and DA is that positive relevance is a symmetric relation (see Spohn, 2012, ch. 6):

$$\textit{If}\ \tau(C|A) > \tau(C|\bar{A}), \text{then}\ \ \tau(A|C) > \tau(A|\bar{C}) \tag{16}$$

Therefore, if A is positively relevant to C, and it is acceptable to use the conditional in forward inferences from A to C, then C also provides support for A.

Moreover, as Spohn (2013, p. 1092) points out, it also holds that:

$$\text{If}\, A\, \text{is positively relevant to}\, C, \text{then} \sim A\, \text{is positively relevant to} \sim C \tag{17}$$

This, together with Eq. 16, yields contraposition:

$$\text{If}\, A\, \text{is positively relevant to}\, C, \text{then} \sim C\, \text{is positively relevant to} \sim A \tag{18}$$

Equation 16 underwrites acceptable instances of AC, Eq. 17 underwrites acceptable instances of DA, and Eq. 18 underwrites acceptable instances of MT.[8]

Finally, as Spohn (2013, p. 1093) points out, these symmetrical relevance relations make room for explaining the varying degrees of endorsement for these four inferences, because although the relations run in both directions they need not do so to the same degrees (see also Spohn, 2012, p. 112).

However, this only provides us with a rough, qualitative prediction of the results of the experiments on the conditional inference task. But it is definitely on the right track. A typical finding using abstract content and instructions stressing logical necessity is that MP>MT>AC≥DA, and a typical finding using realistic content is that the degrees of endorsement depend on perceived sufficiency and necessity of the antecedent for the consequent (Klauer, Beller, & Hütter, 2010). What these ratings show is that we are not looking for a relation governed by perfect symmetry when modeling the relationship between the antecedent and consequent in conditionals, because then we would end up with the bi-conditional interpretation, wherein MP, MT, AC, and DA should all be fully endorsed to the same degrees. On the other hand, the data do not support the material implication interpretation, whereby MP and MT should be fully endorsed while AC and DA should be fully rejected (Evans & Over, 2004). Instead, what we see is that all four inferences are endorsed, but to different degrees, which requires a relationship between the antecedent and consequent that holds in both directions but to different degrees.

4.1. The logistic regression model

If we are to turn these observations into quantitative predictions, we can exploit the fact that something similar holds for logistic regression. Firstly, it is useful to note that the following fact about linear regression has a counterpart in logistic regression.

Correlation and linear regression are sometimes[9] distinguished by pointing out that the former is symmetric, whereas the latter is asymmetric in the following sense: In the case of a correlation, no distinction is made between dependent and independent variables, whereas it makes a difference which variables are treated as dependent and independent in a regression equation.

To be sure, it is possible to treat Y as a predictor of X instead of treating X as a predictor of Y by using Table 2, where 's_Y' is the sample standard deviation, 's_{XY}' is the sample covariance, 'r_{XY}' is the sample correlation coefficient, and '$\bar{x}$' is the sample mean.

But the regression lines, to which the scatter plot will be fitted, will differ depending on whether X is treated as a predictor of Y or Y is treated as a predictor of X. It turns out that something similar holds for logistic regression, when the independent variable is also a binary variable.[10] With this in mind, we now turn to the asymmetry between when X is used as a predictor of Y and Y is used as a predictor of X in logistic regression, as exhibited in Table 3. As we notice, the slopes are identical,[11,12] but the intercepts differ.

Accordingly, we have now reached a point, where we are able to see that the logistic regression equations give us a model of a predictor relationship that has the desired property of a relation that holds in both directions but to different degrees, which we observed above would be useful in modeling the degrees of endorsement of MP, MT, AC, and DA. Exploiting this fact, the following equations can be formulated, where the consequent ($C = \{Y = 1\}$, non-$C = \{Y = 0\}$) is also used as a predictor of the antecedent ($A = \{X = 1\}$ and non-$A = \{X = 0\}$):

$$\text{(MP)} \quad P(Y = 1|X = 1) = \frac{1}{1 + e^{-(b_0+b_1)}} \tag{19}$$

$$\text{(AC)} \quad P(X = 1|Y = 1) = \frac{1}{1 + e^{-(b_0^*+b_1)}} \tag{20}$$

Table 2
Linear Regression

	X as a Predictor of Y	Y as a Predictor of X
Slope	$b_1 = r_{XY} \cdot \frac{s_Y}{s_X} = \frac{s_{XY}}{s_X^2}$	$b_1{}^* = r_{XY} \cdot \frac{s_X}{s_Y} = \frac{s_{XY}}{s_Y^2}$
Intercept	$b_0 = \bar{y} - b_1 \cdot \bar{x}$	$b_0{}^* = \bar{x} - b_1{}^* \cdot \bar{y}$

Table 3
Logistic Regression

	X as a Predictor of Y	Y as a Predictor of X
Intercept	$e^{b_0} = \frac{P(Y=1\|X=0)}{P(Y=0\|X=0)}$	$e^{b_0^*} = \frac{P(X=1\|Y=0)}{P(X=0\|Y=0)}$
"Slope"	$e^{b_1} = \frac{P(Y=1,X=1)}{P(Y=0,X=1)} \cdot \frac{P(Y=0,X=0)}{P(Y=1,X=0)}$	

$$\text{(DA)} \quad P(Y=0|X=0) = \frac{1}{1+e^{b_0}} \tag{21}$$

$$\text{(MT)} \quad P(X=0|Y=0) = \frac{1}{1+e^{b_0^*}} \tag{22}$$

As we shall see later, these equations have a range of nice predictions. In section 3.2 we already noticed the close relationship between logistic regression, which has a logged odds format, and two-sided ranking functions when a logarithmic base of e^{-1} is chosen. It is now possible to make the parallel even closer by considering Eqs. 19–22 under a different light. In their logged odds format they take the following form:

$$\text{(MP)} \quad \ln\left(\frac{P(Y=1|X=1)}{P(Y=0|X=1)}\right) = b_0 + b_1 \tag{23}$$

$$\text{(AC)} \quad \ln\left(\frac{P(X=1|Y=1)}{P(X=0|Y=1)}\right) = b_0^* + b_1 \tag{24}$$

$$\text{(DA)} \quad \ln\left(\frac{P(Y=0|X=0)}{P(Y=1|X=0)}\right) = -b_0 \tag{25}$$

$$\text{(MT)} \quad \ln\left(\frac{P(X=0|Y=0)}{P(X=1|Y=0)}\right) = -b_0^* \tag{26}$$

However, since the following holds:

$$\ln\left(\frac{P(Y=1|X=1)}{P(Y=0|X=1)}\right) = \log_{\frac{1}{e}}\left(\frac{P(Y=0|X=1)}{P(Y=1|X=1)}\right) = \tau(C|A)$$

$$\ln\left(\frac{P(X=1|Y=1)}{P(X=0|Y=1)}\right) = \log_{\frac{1}{e}}\left(\frac{P(X=0|Y=1)}{P(X=1|Y=1)}\right) = \tau(A|C)$$

$$\ln\left(\frac{P(Y=0|X=0)}{P(Y=1|X=0)}\right) = \log_{\frac{1}{e}}\left(\frac{P(Y=1|X=0)}{P(Y=0|X=0)}\right) = \tau(\bar{C}|\bar{A})$$

$$\ln\left(\frac{P(X=0|Y=0)}{P(X=1|Y=0)}\right) = \log_{\frac{1}{e}}\left(\frac{P(X=1|Y=0)}{P(X=0|Y=0)}\right) = \tau(\bar{A}|\bar{C})$$

we now see that:

$$\text{(MP)} \qquad \tau(C|A) = b_0 + b_1 \tag{27}$$

$$\text{(AC)} \qquad \tau(A|C) = b_0^* + b_1 \tag{28}$$

$$\text{(DA)} \qquad \tau(\bar{C}|\bar{A}) = -b_0 \tag{29}$$

$$\text{(MT)} \qquad \tau(\bar{A}|\bar{C}) = -b_0^* \tag{30}$$

Furthermore, Table 3 can be reformulated on the basis of two-sided ranking functions as shown in Table 4.

Table 4 makes the parametrization much more perspicuous than Table 3. In the case of b_0 and b_0*, we are dealing with a measure of our belief in the consequent when the predictor takes the value "false," whereas the b_1 parameter quantifies the *relevance* of the predictor for the consequent. We moreover observe that in spite of the fact that the absolute magnitudes of $\tau(C|A)$ and $\tau(C|\bar{A})$ may diverge from the magnitudes of $\tau(A|C)$ and $\tau(A|\bar{C})$, respectively, the differences in these pairs stay identical and, therefore, the b_1 parameter stays the same no matter from which direction we view the predictor relationship. However, as the intercepts differ, situations may arise where C (taking the place as a conclusion with A as a premise) should be endorsed to a greater degree than A (when A is the conclusion and C is a premise).

To explain these parallels we are observing between logistic regression and two-sided ranking functions, it suffices to note that:

$$b_0 + b_1 = \ln\left(\frac{P(Y=1|X=0)}{P(Y=0|X=0)}\right) + \ln\left(\frac{\frac{P(Y=1|X=1)}{P(Y=0|X=1)}}{\frac{P(Y=1|X=0)}{P(Y=0|X=0)}}\right) = \ln\left(\frac{P(Y=1|X=1)}{P(Y=0|X=1)}\right)$$

But, of course:

$$\ln\left(\frac{P(Y=1|X=1)}{P(Y=0|X=1)}\right) = \log_{\frac{1}{e}}\left(\frac{P(Y=0|X=1)}{P(Y=1|X=1)}\right) = \tau(C|A)$$

Moreover, something similar holds for Eqs. 28–30. In other words, it turns out that Eqs. 19–22 can be derived from probabilistic transformations of two-sided ranking functions once a logarithmic base of e^{-1} is chosen. This observation is extremely useful, because it implies that we can use Eqs. 19–22 to derive precise, quantitative predictions for what had to remain qualitative predictions in Spohn (2013). In section 5 we will see exactly how rich these predictions turn out to be.

Table 4
Translation of Table 3 into ranking functions

	X as a Predictor of Y	Y as a Predictor of X
Intercept	$b_0 = \tau(C|\bar{A})$	$b_0^* = \tau(A|\bar{C})$
"Slope"	$b_1 = \tau(C|A) - \tau(C|\bar{A}) = \tau(A|C) - \tau(A|\bar{C})$	

4.2. Modeling the conditional inference task

The model provided in section 4.1 has three free parameters. We will now see how one can introduce qualitative constraints on the values assigned to the estimated parameters.

To do so, it is useful to keep the theoretical background in mind. In the psychology of reasoning there has been a focus on the influence of *disablers* and *alternative antecedents* on conditional reasoning, which has, *inter alia*, been used to measure the influence of content on deductive reasoning. Disablers are conditions that prevent the consequent from obtaining even when the antecedent obtains, and alternative antecedents are conditions other than the antecedent that are sufficient for bringing about the consequent. Therefore, if we take the conditional "if the key is turned, the car will start," "the battery is dead" would be a disabler and "the car has been hot-wired" would be an alternative antecedent.

There is an experimental paradigm, going back to Thompson (1994) and Cummins (1995), which has studied how the endorsement rates of MP, MT, AC, and DA are affected by changes in the perceived sufficiency and necessity of the antecedent for the consequent. Such changes are induced by manipulating the availability of disablers and alternative antecedents. The general finding is that endorsement rates of AC and DA decrease with the availability of alternative antecedents, and the endorsement rates of MP and MT decrease with the availability of disablers (Politzer & Bonnefon, 2006). Subsequent models of conditional reasoning have focused on integrating a component which takes activation of memory traces of disablers and alternative antecedents into account (Cummins, 2010; De Neys, 2010). Moreover, studies based on means–end relations, permission, precaution, promises, tips, warnings, threats, temporal relations, and obligations have shown that the phenomenon generalizes beyond causal inferences (Beller, 2008; Politzer & Bonnefon, 2006, p. 497; Verbrugge, Dieussaert, Schaeken, Smessaert, H. & William, 2007; see also Oaksford & Chater, 2010b).

Due to this theoretical background, it is a nice feature of our model that it is able to integrate the influence of disablers and alternative antecedents. Indeed, the illumination that the present account brings to this issue goes beyond this, because through Spohn's (2012: ch. 6) notion of sufficient and necessary reasons, we are able to make sense of the talk in the psychological literature about *degrees* of perceived sufficiency and necessity by pointing out that the former can be cashed out in terms of how far above 0 $\tau(C|A)$ is and the latter can be cashed out in terms of how far below 0 $\tau(C|\bar{A})$ is.[13]

The way in which the model integrates the influence of disablers and alternative antecedents is by the intended interpretation of its parameters, as outlined in Table 3. A natural assumption is: (a) that the *presence of disablers* has the effect of *increasing* $P(X = 1,\ Y = 0)$ (i.e., "$P(X = 1,\ Y = 0)\uparrow$") and *decreasing* $P(X = 1,\ Y = 1)$ (i.e., "$P(X = 1, Y = 1)\downarrow$"); and (b) that *the presence of alternative antecedents* has the effect of $P(X = 0, Y = 1)\uparrow$ and $P(X = 0, Y = 0)\downarrow$. According to Table 3, the presence of disablers should thus have the effect of $b_0{*}\uparrow$ and $b_1\downarrow\downarrow$, and the presence of alternative antecedents should have the effect of $b_0\uparrow\uparrow$, $b_0{*}\uparrow$, and $b_1\downarrow\downarrow$. This introduces a qualitative constraint on the values assigned to the parameters of the model to ensure that it does not merely fit the data due to the flexibility generated by its free parameters.

When it comes to modeling the conditional inference task, it should be noted that Klauer et al. (2010) employ two versions of the task. On the one hand, there is a baseline condition, where the conditional rule has been removed and the participants are evaluating the conclusion merely on the basis of the minor premise and their background knowledge (i.e., MP_R: $p \therefore q$; MT_R: $\neg q \therefore \neg p$; AC_R: $q \therefore p$; DA_R: $\neg p \therefore \neg q$, where the subscript stands for "reduced"). On the other hand, there is a rule-present condition, where the conditional rule has been added as a major premise, as in the original version of the task.

A natural assumption is that the baseline condition makes the participants access their conditional beliefs to assign probabilities to the conclusion, given the minor premise. Accordingly, their performance in this condition can be modeled by Eqs. 19–22, which depict the conditional probability of the conclusion given the minor premise as determined by two-sided ranking functions representing their conditional beliefs. On the basis of Spohn's relevance approach, it is moreover natural to suppose that the presence of the conditional rule in the original version of the task leads to an increase in the perceived relevance of the antecedent for the consequent, which amounts to an increase in the b_1 parameter. However, as b_1 can increase in several ways, it seems most natural that it takes the form of an increase in $P(Y = 1, X = 1)$, which is counterbalanced by a decrease in $P(Y = 0, X = 1)$ to ensure that we end up with a probability distribution respecting the axioms of the probability calculus. According to Table 3, these changes end up having the effect of $b_0{*}\downarrow$ and $b_1\uparrow\uparrow$, which affects the conditional probabilities calculated by Eqs. 19–22 as follows: MP$\uparrow\uparrow$, MT$\uparrow$, AC$\uparrow$.

If we abstract from a minor increase to DA in figures 4 and 5 in Klauer et al. (2010), which was not consistent across all experiments, this prediction of the effect of the presence of the rule in the conditional inference task fits well with the pattern of results reported in that paper.[14]

According to the Bayesian alternative developed by Oaksford and Chater, which was examined in Klauer et al. (2010), the presence of the rule is to be modeled by $e(D)\downarrow$ in the following equations, which are used for modeling the baseline condition:

$$\text{(MP)} \qquad P(q|p) = 1 - e(D) \tag{31}$$

$$\text{(MT)} \qquad P(\neg p|\neg q) = \frac{1 - b(D) - a(D)e(D)}{1 - b(D)} \tag{32}$$

$$\text{(AC)} \qquad P(p|q) = \frac{a(D)(1 - e[D])}{b(D)} \tag{33}$$

$$\text{(DA)} \qquad P(\neg q|\neg p) = \frac{1 - b(D) - a(D)e(D)}{1 - a(D)} \tag{34}$$

a(D) is the perceived probability of p events for the content D.[15] $b(D)$ is the perceived probability of q events for the content D. Finally, $e(D)$ is the exceptions parameter or the conditional probability of *not-q* given p for the content D.

As H. Singmann & K. C. Klauer (personal communication) have rightly pointed out, Eqs. 19–22 and 31–34 should be mathematically equivalent to the extent to which both models are reparametrizations of the joint probability distribution. However, they are based on different semantics of the conditional, so they should model the presence of the rule differently. In particular, relevance considerations should not play a role in the Oaksford & Chater model as they are not part of the suppositional theory of conditionals on which it is based (Bennett, 2003; Edgington, 1995). Moreover, the suppositional theory of conditionals renders AC and DA invalid (Evans & Over, 2004, p. 45). In contrast, relevance considerations should play a role in the logistic regression model and it is based on a theory that renders MP, MT, AC, and DA acceptable while allowing the endorsement of their respective conclusions to take different degrees. With a bit of calculation, Eqs. 31–34 can be rewritten as follows:

$$\text{(MP)} \qquad \frac{P(X=1, Y=1)}{P(X=1, Y=1) + P(X=1, Y=0)} \tag{35}$$

$$\text{(MT)} \qquad \frac{P(Y=0, X=0)}{P(Y=0, X=0) + P(Y=0, X=1)} \tag{36}$$

$$\text{(AC)} \qquad \frac{P(X=1, Y=1)}{P(X=1, Y=1) + P(X=0, Y=1)} \tag{37}$$

$$\text{(DA)} \qquad \frac{P(X=0, Y=0)}{P(X=0, Y=0) + P(X=0, Y=1)} \tag{38}$$

Supposedly, the decrease in the exceptions parameter, $e(D)$, in the presence of the conditional rule takes the form of $P(Y = 0, X = 1)\downarrow$. The result is MP↑ and MT↑. However, if the axioms of the probability calculus are to be satisfied after this change has occurred, then $P(Y = 0, X = 1)\downarrow$ must be counterbalanced by $P(Y = 1, X = 1)\uparrow$, which yields the same result as above, to wit: MP↑↑, MT↑, AC↑. Yet, it is quite puzzling why a model based on a semantics of the conditional that renders AC invalid should end up predicting that the presence of the conditional rule leads to an increase in AC. This poses a dilemma for the Oaksford and Chater model: Either predictions should be tolerated that are in accordance with the underlying theory at the cost of depicting the participants as probabilistically incoherent, or the model should be allowed to make a prediction that is invalid according to its underlying theory.

5. Deriving further predictions

In deriving further predictions from our model, it is useful to return to the specification of the notions of reason and relevance from section 2.2, extend it to cover all of the cases of A being a reason against C, and substitute regression weights for two-sided ranking functions:

$$\boldsymbol{A \text{ is positively relevant to } C \text{ iff } b_0 + b_1 > b_0} \tag{39}$$

$$A \text{ is a supererogatory reason for } C \text{ iff } b_0 + b_1 > b_0 > 0$$

$$A \text{ is a sufficient reason for } C \text{ iff } b_0 + b_1 > 0 \geq b_0$$

$$A \text{ is a necessary reason for } C \text{ iff } b_0 + b_1 \geq 0 > b_0$$

$$A \text{ is an insufficient reason for } C \text{ iff } 0 > b_0 + b_1 > b_0$$

$$\boldsymbol{A \text{ is irrelevant to } C \text{ iff } b_0 + b_1 = b_0} \tag{40}$$

$$\boldsymbol{A \text{ is negatively relevant to } C \text{ iff } b_0 > b_0 + b_1} \tag{41}$$

$$A \text{ is a supererogatory reason against } C \text{ iff } 0 > b_0 > b_0 + b_1$$

$$A \text{ is a sufficient reason against } C \text{ iff } b_0 \geq 0 > b_0 + b_1$$

$$A \text{ is a necessary reason against } C \text{ iff } b_0 > 0 \geq b_0 + b_1$$

$$A \text{ is an insufficient reason against } C \text{ iff } b_0 > b_0 + b_1 > 0$$

As we notice, A is *positively relevant* to C whenever $b_1 > 0$, A is *irrelevant* to C whenever $b_1 = 0$, and A is *negatively relevant* to C whenever $b_1 < 0$. Using this observation and the inequalities in Eqs. 39–41, it is possible to derive predictions about the degrees of endorsement of MP, MT, AC, and DA for different types of reason relations.

5.1. Sufficiency and necessity

A is a sufficient reason for C:

$$b_0 + b_1 > 0 \leftrightarrow \frac{1}{1 + e^{-(b_0+b_1)}} > \frac{1}{2} \leftrightarrow \text{MP} > \frac{1}{2}$$

$$b_0 \leq 0 \leftrightarrow \frac{1}{1 + e^{b_0}} \geq \frac{1}{2} \leftrightarrow \text{DA} \geq \frac{1}{2}$$

As we have already noted, the degree of perceived sufficiency can be experimentally manipulated through disablers, which increase $P(X = 1, Y = 0)$ and decrease $P(X = 1, Y = 1)$. Using Table 3, we can then see that decreasing the perceived sufficiency through the presence of disablers should have the following impact on the model's parameters: $b_1\downarrow\downarrow$, $b_0{*}\uparrow$. As a result, the presence of disablers should have the effect of decreasing the endorsement of MP:

$$\frac{1}{1+e^{-(b_0+b_1)}} > \frac{1}{1+e^{-(b_0+b_1-a)}} \quad \text{for } a > 0$$

and MT:

$$\frac{1}{1+e^{b_o^*}} > \frac{1}{1+e^{b_o^*+a}} \quad \text{for } a > 0$$

A is a necessary reason for C:

$$b_0 + b_1 \geq 0 \leftrightarrow \frac{1}{1+e^{-(b_0+b_1)}} \geq \frac{1}{2} \leftrightarrow \text{MP} \geq \frac{1}{2}$$

$$b_0 < 0 \leftrightarrow \frac{1}{1+e^{b_0}} > \frac{1}{2} \leftrightarrow \text{DA} > \frac{1}{2}$$

As we have already noted, the degree of necessity can be experimentally manipulated through alternative antecedents, which increase $P(X = 0, Y = 1)$ and decrease $P(X = 0, Y = 0)$. Using Table 3, we can then see that decreasing the perceived necessity through the presence of alternative antecedents should have the following impact on the model's parameters: $b_1\downarrow\downarrow$, $b_0\uparrow\uparrow$, $b_0{*}\uparrow$. As a result, the presence of alternative antecedents should have the effect of decreasing endorsements of DA:

$$\frac{1}{1+e^{b_0}} > \frac{1}{1+e^{b_0+a}} \text{for } a > 0$$

and AC:

$$\frac{1}{1+e^{-\left(b_0^*+b_1\right)}} > \frac{1}{1+e^{-\left(b_0^*+b_1-a_1+a_2\right)}} \text{for } a_1 > a_2 > 0$$

An effect of disablers on MP and MT and of alternative antecedents on AC and DA constitutes one of the major findings in the field, as Politzer and Bonnefon (2006, p. 486) say in the quote below with reference to two well-known experimental paradigms that go back to the work of Byrne (1989), Thompson (1994), and Cummins (1995):

> Thus, the two experimental paradigms concur to what we call here the Core Pattern of results: Disabling conditions defeat the conclusions of MP and MT (but usually not the

conclusions of DA and AC) and alternative conditions defeat the conclusion of DA and AC (but usually not the conclusions of the valid MP and MT). This Core Pattern is endorsed by most if not all researchers in the field, and, apart from the occasional breach (see in particular Markovits and Potvin, 2001), has never been seriously questioned. (Politzer & Bonnefon, 2006, p. 486)

It is thus pleasing to note that the present model is capable of delivering this prediction.

5.2. Possible exceptions

However, it should also be noted that the changes noted above to the parameters of the model further predict that the presence of disablers should lead to a decrease in AC and that the presence of alternative antecedents should lead to a decrease in MT. However, such effects are said usually not to occur in the quote above. Figuring out why the participants fail to comply to these two predictions, whose reasonableness can easily be demonstrated by Venn diagrams, is an important topic for further inquiry. This is accentuated by the fact that the same predictions can be derived from Oaksford & Chater's model once our way of modeling the presence of disablers and alternative antecedents is plotted into Eqs. 35–38.

To illustrate the predictions, consider the following Venn diagrams (Fig. 1).

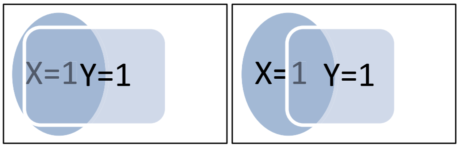

Fig. 1. Venn diagrams illustrating the AC disabler prediction.

We now see on the picture to the right that after more disablers have been added, the proportion of the $Y = 1$ event taken up by the $X = 1$ event has shrunken and the possibility of alternative antecedents as the cause of the $Y = 1$ event should now be attributed a greater weight than before. Of course, this effect only occurs if $Y = 1$ is not a subset of $X = 1$. That is to say, the effect disappears if there are no alternative antecedents and $X = 1$ has a perfect degree of necessity for $Y = 1$, where $P(X = 0, Y = 1) = 0$.

Similarly, the MT effect also disappears if $X = 1$ is a subset of $Y = 1$ and $X = 1$ has a perfect degree of sufficiency for $Y = 1$ (Fig. 2).

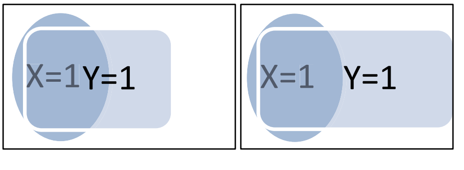

Fig. 2. Venn diagrams illustrating the MT alternative antecedent prediction.

What happens when we add more alternative antecedents to the picture is that the $Y = 1$ event grows and the $Y = 0$ event shrinks, as shown on the picture to the right. As a result, the $X = 1 \cap Y = 0$ event now takes up a larger portion of the $Y = 0$ event, and $P(X = 0|Y = 0)$ has become less likely than before.

To get an intuitive grasp of the $AC_{disabler}$ prediction, it is perhaps best to consider the case of causal relations. If the causal relation between A and C is used to infer the occurrence of the cause based on the occurrence of the effect, then disablers that weaken this causal relation should presumably also have an impact on such abductive inferences. Thus, it might, for instance, be most likely that the oldest brother, Tim, took care of the siblings' elderly, sick mother given that Tim has always had a strong sense of responsibility. But if told that Tim is himself ill and weak at the moment, the possibility that one of the other siblings acted as a caretaker should be assigned a greater weight than it was before. Consequently, we should be less inclined to infer that Tim was the one who took care of his sick mother after learning that Tim himself has turned ill than we were before.

Due to the difficulty with processing negations, it is a bit harder to get one's head around the MT prediction. Therefore, to ease the processing demands, lexicalized negations (e.g., losing) can be used as a substitute of explicit negations (not winning).[16] Let's suppose then that $Y = 1$ is the event that the blue (underwater rugby) team wins and that $X = 1$ is the event at which the blue captain is present. Then the conditional rule under consideration is "if the blue captain is present, the blue team will win." As the prediction only holds for less than perfect degrees of sufficiency, it is required that a disabler such as "the blue captain is present but distracted by an important, upcoming exam" is active some of the time. This gives us the following inference:

If the blue captain is present, blue team will win
The blue team has lost

∴The blue captain was absent

Now the point is that this inference should seem more likely before further alternative antecedents are added to the picture, for example, that the blue team has recently acquired the star player, Zack, who is fully capable of securing a victory, as a replacement for the blue captain, when he is absent. The reason is that explaining the loss by the absence of the blue captain now also requires that he was not replaced by Zack. That is to say, it is now rarer that the blue team loses and the proportion of the cases where the loss is due to the absence of the blue captain has now shrunken. As a result, the possibility that the loss was due to the disabler that the blue captain was present but distracted by an important, upcoming exam (and therefore played terribly) should now be given a greater weight than it was before.

Such considerations indicate that these predictions are normatively correct. But the participants are apparently not sensitive to them—which is perhaps not surprising given that their correctness appears to have gone by unnoticed in the psychological literature as well.[17]

5.3. Cases and distinctions not covered by other theories

A is a supererogatory reason for C:

$$b_0 + b_1 > 0 \leftrightarrow \frac{1}{1 + e^{-(b_0+b_1)}} > \frac{1}{2} \leftrightarrow \text{MP} > \frac{1}{2}$$

$$b_0 > 0 \leftrightarrow \frac{1}{1 + e^{b_0}} < \frac{1}{2} \leftrightarrow \text{DA} < \frac{1}{2}$$

In contrast with sufficient and necessary reasons, supererogatory and insufficient reasons are normally not conceptually distinguished in the experimental literature. To experimentally manipulate supererogatory reasons, not only the presence of alternative antecedents should be manipulated, but also their obtainance. Therefore, whereas a necessary reason would require the absence of alternative antecedents and, thus, require high ratings of AC and DA, a supererogatory reason would require the presence and obtainance of alternative antecedents and thus require low ratings for AC and DA. Necessary and supererogatory reasons can moreover be distinguished by the prediction displayed above that, whereas DA > 0.5 and MP ≥ 0.5 hold for necessary reasons, and DA < 0.5 and MP > 0.5 hold for supererogatory reasons.

A is an insufficient reason for C:

$$b_0 + b_1 < 0 \leftrightarrow \frac{1}{1 + e^{-(b_0+b_1)}} < \frac{1}{2} \leftrightarrow \text{MP} < \frac{1}{2}$$

$$b_0 < 0 \leftrightarrow \frac{1}{1 + e^{b_0}} > \frac{1}{2} \leftrightarrow \text{DA} > \frac{1}{2}$$

To experimentally manipulate insufficient reasons, not only the presence of disablers should be manipulated, but also their obtainance. Therefore, whereas a sufficient reason would require the absence of disablers and, thus, require high ratings of MP and MT, an insufficient reason would require low ratings of MP and MT (which is known as *the suppression effect* in the psychological literature).[18] Sufficient and insufficient reasons can moreover be distinguished by the prediction that MP > 0.5 and DA ≥ 0.5 for sufficient reasons, whereas MP < 0.5 and DA > 0.5 for insufficient reasons.

A is irrelevant to C: Since the prevailing semantics of conditionals in the psychology of reasoning do not take the dimension of relevance into account, predictions of patterns of conditional reasoning under manipulations of relevance hold the prospect of being *unique* to the present model.[19]

In the case of irrelevance, conditionalizing on the antecedent will not affect the probability of the consequent. It thus holds that:

$$e^{b_0^*} = \frac{P(X=1|Y=0)}{P(X=0|Y=0)} = \frac{P(X=1)}{P(X=0)} \leftrightarrow b_0^* = ln\left(\frac{P(X=1)}{P(X=0)}\right)$$

$$e^{b_0} = \frac{P(Y=1|X=0)}{P(Y=0|X=0)} = \frac{P(Y=1)}{P(Y=0)} \leftrightarrow b_0 = ln\left(\frac{P(Y=1)}{P(Y=0)}\right)$$

This observation, together with our earlier observation that $b_1 = 0$ for irrelevance, can be used to derive predictions for content manipulations of the prior probability of the antecedent and the consequent:

Table 5
Predictions for Irrelevance cases

	$P(C) > 0.5$	$P(C) = 0.5$	$P(C) < 0.5$
$P(A) > 0.5$	AC > MT, MP > DA	AC > MT, MP = DA	AC > MT, MP < DA
$P(A) = 0.5$	AC = MT, MP > DA	AC = MT, MP = DA	AC = MT, MP < DA
$P(A) < 0.5$	AC < MT, MP > DA	AC < MT, MP = DA	AC < MT, MP < DA

Note: AC > MT: the degree of endorsement of AC >the degree of endorsement of MT.

The following example illustrates the approach for $P(A) > 0.5$, $P(C) > 0.5$:

$$P(X=1) > 0.5 \leftrightarrow b_0^* > 0 \leftrightarrow \frac{1}{1+e^{-b_0^*}} > \frac{1}{1+e^{b_0^*}} \leftrightarrow \text{AC} > \text{MT}$$

$$P(Y=1) > 0.5 \leftrightarrow b_0 > 0 \leftrightarrow \frac{1}{1+e^{-b_0}} > \frac{1}{1+e^{b_0}} \leftrightarrow \text{MP} > \text{DA}$$

What the predictions in Table 5 show is that when the antecedent is irrelevant to (or statistically independent of) the consequent, the model predicts that the four inferences coincide with what one would arrive at by using the prior probability of the conclusion while ignoring the probability of the premise. Intuitively, this seems exactly right.

Moreover, it holds in general that:

Table 6
Further predictions for the irrelevance case

$P(A) > P(C)$	MT < DA, AC > MP
$P(A) = P(C)$	MT = DA, MP = AC
$P(A) < P(C)$	MT > DA, AC < MP

To illustrate, if $P(A) = P(C)$, then:

$$\frac{1}{1+e^{b_0^*}} = \frac{1}{1+e^{b_0}} \leftrightarrow \text{MT} = \text{DA}$$

$$\frac{1}{1+e^{-b_0^*}} = \frac{1}{1+e^{-b_0}} \leftrightarrow \mathrm{AC} = \mathrm{MP}$$

Again, the predictions in Table 6 are also what one would expect for cases where the antecedent is irrelevant to the consequent, insofar as the probabilities of the conclusions coincide with their prior probabilities.

Finally, it can be observed that MP = 1 − DA and AC = 1 − MT as $b_1 = 0$ in the case of irrelevance:

$$\frac{1}{1+e^{-b_0}} = 1 - \frac{1}{1+e^{b_0}}$$

$$\frac{1}{1+e^{-b_0^*}} = 1 - \frac{1}{1+e^{b_0^*}}$$

A is a reason against C:

One way in which to view cases where the antecedent is a reason *against* the consequent is to view them as negating the consequent of cases of positive relevance. As a result, if A is a sufficient reason *for* C, then A is, *ipso facto*, also a sufficient reason *against* ~C. To see that this is so, it is easiest to use the probabilistic version of Eqs. 39 and 41:

$$P(Y=1|X=1) > 0.5 \leftrightarrow P(Y=0|X=1) < 0.5$$

$$0.5 \geq P(Y=1|X=0) \leftrightarrow P(Y=0|X=0) \geq 0.5$$

Thus:

$$P(Y=1|X=1) > 0.5 \geq P(Y=1|X=0) \leftrightarrow P(Y=0|X=0) \geq 0.5 > P(Y=0|X=1)$$

Similarly, it can be shown that if A is a supererogatory reason *for* C, then A is a supererogatory reason *against* ~C; if A is a necessary reason *for* C, then A is a necessary reason *against* ~C; and if A is an insufficient reason *for* C, then A is an insufficient reason *against* ~C. To emphasize this connection it may be useful to reformulate Eq. 41, so it becomes perspicuous that if the relations in Eq. 39 hold for C, then the following holds for its negation:

$$\textbf{\textit{A} is negatively relevant to} \neg \boldsymbol{C} \textbf{ iff } -\boldsymbol{b_0} > -\boldsymbol{b_0} - \boldsymbol{b_1} \tag{42}$$

$$A \text{ is a supererogatory reason against} \neg C \text{ iff } 0 > -b_0 > -b_0 - b_1$$

$$A \text{ is a sufficient reason against} \neg C \text{ iff } -b_0 \geq 0 > -b_0 - b_1$$

$$A \text{ is a necessary reason against} \neg C \text{ iff } -b_0 > 0 \geq -b_0 - b_1$$

$$A \text{ is an insufficient reason against} \neg C \text{ iff } -b_0 > -b_0 - b_1 > 0$$

What this shows is that if we have a reason, A, *against* C that takes one of the four forms in Eq. 41, then the predictions specified for the corresponding positive relevance relation will hold for when A is taken as a reason *for* $\sim C$ and, *vice versa*.

6. Comparison with the Ramsey test

After this inspection of some of the predictions that can be derived from the logistic regression model, it is time to return to the more theoretical side. On the basis of equating $P(\text{if } p, q)$, or $Acc(\text{if } p, q)$, with $P(q|p)$ (cf. endnote 4), the Ramsey test is used in the suppositional theory to estimate the probability of natural language conditionals (Evans & Over, 2004). What it requires is that the subject adds p to his knowledge base and estimates $P(q)$ on this basis.

However, how exactly this is carried out is not entirely clear, as Over, Hadjichristidis, Evans, Handley, and Sloman (2007) point out:

> Explaining how the Ramsey test is actually implemented—by means of deduction, induction, heuristics, causal models, and other processes—is a major challenge, in our view, in the psychology of reasoning. (p. 63).

That is to say, the Ramsey test does not explain how $P(q)$ is determined once p has been added to the subject's knowledge base. Here, the model allows us to come up with the following elegant suggestion: upon adding the antecedent to our belief set, its weight as a predictor of the consequent is used to compute the posterior probability. Moreover, as the parameters of the model could be expressed in terms of two-sided ranks in Eqs. 27–30, the agent's conditional beliefs are being accessed in carrying out these computations.

Once this computational task has been formulated, it becomes possible to start theorizing about the cognitive processes carrying out the computations (e.g., fast and frugal heuristics) and the mediating factors which could influence this computation of the posterior probability in virtue of the regression weights. In particular, knowledge about causal models may influence the assigned weight and judgments of a hypothesis' virtues as an explanation may influence the weight in the case of use of Y as a predictor of X.

Moreover, the theoretical importance of this suggestion about the Ramsey test can be explicated in the following manner. According to Evans and Over (2004) and Evans (2007), "if then" is a linguistic device that is used to simulate possibilities by activating a mental algorithm that makes us probe our background knowledge according to the Ramsey test.[20] However, although it makes a great deal of sense to say that simulating possibilities is useful from an evolutionary perspective, simulating possibilities is not by itself evolutionarily useful when the antecedent is irrelevant to the consequent. This suggests that the dimension of relevance adds to the idea of the function of the conditional as consisting of simulating possibilities. More generally, conditionals can be thought as serving an important communicative function in sharing knowledge about predictor relationships, which is seen in particular with indicative conditionals containing the predictive modal "will" as in "if it rains the match *will* be canceled" (cf. Dancygier, 1998; Dancygier & Sweetser, 2005).[21]

From this perspective, one of the main points of simulating possibilities can be seen as consisting of evaluating whether the information on offer codifies useful information that the subject can adapt to improve his/her ways of coping with the uncertain environment. Therefore, when a speaker states an indicative conditional, the hearer can be seen as using "if then" as a guide that possibilities are to be simulated, because the consequent is to be evaluated under the supposition of the antecedent (in agreement with Evans and Over [2004]). Yet, the evolutionary point of this exercise consists of its being a way of evaluating whether useful predictive information is being shared. Accordingly, the hearer should view it as a failure if the antecedent is irrelevant and leaves the probability of the consequent unchanged. We thus begin to see how relevance considerations may enter into this process of mental simulation in accordance with the suggestion of the computational task involved in performing the Ramsey test outlined above.

Indeed, it is possible to go further than this and establish a link to Rescorla and Wagner's work on classical conditioning by saying that the information shared by indicative conditionals containing the predictive modal "will" is a linguistic counterpart of the kind of information acquired in classical conditioning. The classification that Granger and Schlimmer (1986, p. 150) attribute to Rescorla in the following quote corresponds exactly to the probabilistic version of Spohn's (2013) analysis of positive relevance, negative relevance, and irrelevance:

> Experiments explicitly aimed at exploring the space of possible contingencies led Rescorla to form the characterization that if $p(\mathrm{US}|\mathrm{CS}) > p(\mathrm{US}|\overline{\mathrm{CS}})$, then excitatory conditioning occurs, and if $p(\mathrm{US}|\mathrm{CS}) < p(\mathrm{US}|\overline{\mathrm{CS}})$, then inhibitory conditioning occurs, and if $p(\mathrm{US}|\mathrm{CS}) = p(\mathrm{US}|\overline{\mathrm{CS}})$, then neither type of conditioning occurs [US = unconditioned stimuli; CS = conditioned stimuli]. (Granger and Schlimmer, 1986, p. 150)

It thus seems that sensitivity to epistemic relevance is a candidate for being a more general feature of our cognitive architecture.

However, it should be noted that although an effect of relevance has been found on the acceptability of conditionals (Douven & Verbrugge, 2012), only a weak effect was found for $P(C|\text{non-}A)$ as a predictor of $P(\text{if } A, C)$ in Over et al. (2007), and no effect was found for $P(C|\text{non-}A)$ as a predictor of $P(\text{if } A, C)$ in Singmann, Klauer, and Over (2014). There are thus unsettled theoretical and empirical questions about how such results bear on the present work on the conditional inference task.

7. Empirical underdetermination?

Before we end, a potential objection needs to be addressed. Once a different translation manual for probabilities into ranks is accepted with a logarithmic base of $a \in (0,1)$, where $a \neq \varepsilon$, there is no *a priori* way of selecting a non-arbitrary value for a from the infinity of possible values. On the face of it, this realization is devastating to any attempt of deriving precise predictions from ranking theory if it implies that any unsuccessful prediction could just be excused by claiming that the wrong logarithmic base had been chosen. This might seem to be a case of radical underdetermination by the empirical evidence. However, such an appearance would be misleading and it turns out that there is a pragmatic solution to this problem, as we shall see.

The first thing to notice is that although there is no *a priori* basis for selecting a logarithmic base other than the infinitesimal base, this does not mean that we are completely without constraints. In particular, one of the main problems with the infinitesimal translation of probabilities into ranks from section 2 is that it seems to fit too poorly with the way in which humans carve up the probability scale by concentrating all our degrees of disbeliefs in probabilities from 0 to $0 + \varepsilon$. This suggests that our choice of a logarithmic base should be constrained empirically. In this context, it is worth noting that Spohn (2013) suggests that it would be possible to align ranking functions with the linguistic qualifiers we use to express our degrees of beliefs. This suggests that independent evidence of the numerical values that ordinary participants associate with verbal expressions of degrees of beliefs should be used in selecting the logarithmic base.

If a logarithmic base of e^{-1} is chosen, then it will be possible for the ranks to spread out more widely over the probability scale, which gives us the following scale (Fig. 3).

$\tau(\cdot\|\cdot)$	-3	-2	-1	0	1	2	3
$P(\cdot\|\cdot)$	$\frac{1}{1+e^3}$ 0.05	$\frac{1}{1+e^2}$ 0.12	$\frac{1}{1+e}$ 0.27	0.5	$\frac{e}{1+e}$ 0.73	$\frac{e^2}{1+e^2}$ 0.88	$\frac{e^3}{1+e^3}$ 0.95

Fig. 3. Two-sided ranking functions with a logarithmic base of e^{-1}.

Incidentally, this scale fits nicely with the following scale, which has already received empirical support (e.g. Witteman & Renooij, 2003, p. 120), and been successfully used for eliciting expert knowledge for Bayesian networks (Van der Gaag, Renooij, Schijf, Elbers, & Loeffen, 2012) (Fig. 4).[22]

However, it is possible that this scale may eventually be replaced by other scales that are better able to capture the linguistic phenomenology of expressing degrees of beliefs. Therefore, the policy that I have adopted in this paper is to use a logarithmic base of e^{-1} for illustrative purposes and be prepared to revise the equations if another grading receives *independent support*. To the extent that such evidence is independent of the performance of the model on the conditional inference task, its calibration by it should not be seen as a question-begging attempt to dodge unpleasant challenges.

The second thing to note is that as far as model fitting goes, it actually does not matter exactly which logarithmic base we select. The reason is that Eqs. 19–22 have

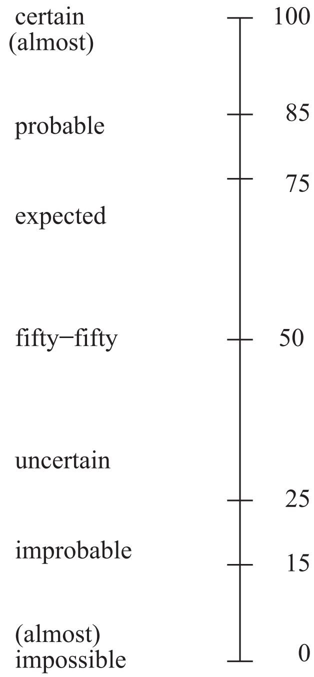

Fig. 4. Verbal-numerical probability scale of Witteman and Renooij (2003).

three parameters that will have to be estimated on the basis of the data. Thus, if the logarithmic base is changed, then the effect will just be to change the order of the magnitude of the estimated regression weights, which merely changes the conventions for interpreting the size of the estimated coefficients. Therefore, the problem of the lack of a principled basis for choosing a logarithmic base will not prevent its use for experimental purposes.

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Notes

1. Important qualification: it should be noted, however, that Leitgeb (forthcoming) is challenging the consensus that one cannot represent full beliefs in probabilistic terms. So a future, an interesting development will be comparisons between ranking

theory and Leitgeb's approach. In Raidl and Skovgaard-Olsen (in review) a first stab is taken.

2. Reference: cf. Spohn (2013).
3. On how to express Spohn's taxonomy of reason relations into probabilities: it should be noted that treating $\tau(C|A) = 0$ as the point of doxastic indifference for two-sided ranking functions commits the theory to treating $P(C|A) = 0.5$ as expressing doxastic indifference, because

$$\tau(C|A) = 0 \approx \log_a\left(\frac{P(Y=0|X=1)}{P(Y=1|X=1)}\right) = 0 \leftrightarrow \frac{P(Y=0|X=1)}{P(Y=1|X=1)}$$
$$= 1 \leftrightarrow P(Y=0|X=1) = P(Y=1|X=1) = 0.5$$

This point appears to have not been fully realized, with Spohn (2012) suggesting that there is no equivalent in the probability calculus for Eqs. 11a–11d. It should moreover be noted that the taxonomy in Eqs. 11a–11d was challenged in Olsen (2014) (Appendix 2).

4. Important qualification: or, rather, strictly speaking, $P(\text{if } A, C) = P(C|A)$ should be replaced by an equation relating the *acceptability* of a conditional to a conditional probability (i.e., $\text{Acc}(\text{if } A, C) = P(C|A)$) due to technical reasons relating to Lewis's triviality results. But usually this subtle difference is not observed in the psychological literature (cf. Douven, 2015: ch. 3).
5. On conditional odds: $O_i = \frac{P(Y=1|X_1,\ldots,X_n)}{1-P(Y=1|X_1,\ldots,X_n)} \quad \text{for } i = 1,\ldots,n.$
6. Reference: see also Politzer and Bonnefon (2006).
7. Acknowledgment: thanks to Karolina Krzyżanowska for pressing me on this issue.
8. Extension: an intriguing possibility is to apply Douven's (2015, ch. 5) account of graded validity to the present statements about the premises providing support for the conclusion in all four inferences (MP, MT, AC, and DA).
9. Reference: Eid et al. (2010) (section 16.6), Howell (1997, ch. 9).
10. Acknowledgment: in discovering this, I was helped by the responses to my inquiry at a forum for statisticians: stats.stackexchange.com/questions/66430.
11. Proof: $e^{b_1} = \frac{\frac{P(Y=1|X=1)}{P(Y=0|X=1)}}{\frac{P(Y=1|X=0)}{P(Y=0|X=0)}} = \frac{P(Y=1|X=1)}{\frac{P(X=1|Y=0)\cdot P(Y=0)}{P(X=1)}} \cdot \frac{P(Y=0|X=0)}{\frac{P(X=0|Y=1)\cdot P(Y=1)}{P(X=0)}} = \frac{P(Y=1,\, X=1)}{P(X=1,\, Y=0)} \cdot \frac{P(Y=0,\, X=0)}{P(X=0,\, Y=1)}$
12. Proof: $e^{b_i^*} = \frac{\frac{P(X=1|Y=1)}{P(X=0|Y=1)}}{\frac{P(X=1|Y=0)}{P(X=0|Y=0)}} = \frac{P(X=1|Y=1)}{\frac{P(Y=1|X=0)\cdot P(X=0)}{P(Y=1)}} \cdot \frac{P(X=0|Y=0)}{\frac{P(Y=0|X=1)\cdot P(X=1)}{P(Y=0)}} = \frac{P(Y=1,\, X=1)}{P(Y=1,\, X=0)} \cdot \frac{P(Y=0,\, X=0)}{P(Y=0,\, X=1)}$
13. Same point in probabilistic terms: in probabilistic terms, the degree of perceived sufficiency can be cashed out in terms of how far above 0.5 $P(Y = 1|X = 1)$ is, and the talk about degrees of perceived necessity can be cashed out in terms of how far below 0.5 $P(Y = 1|X = 0)$ is.
14. Reference: for a more extensive discussion see Olsen (2014: ch. 4).
15. Terminological note: the content variable had to receive a different name from in the original model as we have been using "*C*" to designate the consequent.

16. Reference: cf. Sperber, Cara, and Girotto (1995).
17. Reference: an extended discussion of these exceptions along with their possible connection to the confirmation bias can be found in Olsen (2014, pp. 134–140).
18. Reference: cf. Oaksford and Chater (2010b).
19. Qualification: arguably this does not hold for the rule-free baseline condition, where the Oaksford & Chater model coincides with the present model. However, it would hold for the presence of the conditional rule provided that a way is found of modeling cases, where the latter is characterized by irrelevance or negative relevance (in addition to the positive relevance case considered in section 4.2).
20. Acknowledgment: this useful way of Evans and Over's position is due to Karl Christoph Klauer (personal communication).
21. Qualification: of course, conditionals also serve other important roles, such as contributing to providing explanations and making generalizations. However, these further roles seem closely related to the capacity to share information about predictor relationships. Furthermore, one approach to counterfactuals and fictions would be to view these as domains where we "play" with our understanding of predictor relationships and investigate what we assume would have happened given premises that diverge from our present background beliefs.
22. Permission: this scale is here being reproduced with the kind permission of Silja Renooij.

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